Finite group actions as free resolutions

\[ R = \mathbb{C}[x, y] \]
\[ I = \langle x^3, x^2 y, x y^3, y^3 \rangle \]

\[ M = R/I, \text{ } R\text{-module} \]

\[ R/I \text{ is generated (as } R\text{-module) by } \bar{1} \]

Presentation of \( R/I \):

\[ \begin{array}{cccccc}
\bar{0} & \rightarrow & R/I & \rightarrow & R & \rightarrow \text{d}_1 & \rightarrow \text{d}_2 & \rightarrow \text{ker}(d_2)
\end{array} \]

What is \( \ker(d_2) \)?

\[ d_1(ye_1 - xe_2) = y \cdot x^3 - x \cdot x^2 y = 0 \]

\[ d_2 \left( \begin{array}{ccc}
\bar{3}
\end{array} \right) = \begin{bmatrix}
y & 0 & 0 \\
-x & y & 0 \\
0 & -x & y \\
0 & 0 & -x \\
\end{bmatrix} \]

\[ d_2(f_1) = ye_1 - xe_2 \]

\[ \begin{array}{cccccc}
R^4 & \rightarrow & R^3 & \rightarrow & \text{d}_2 & \rightarrow \text{d}_1 & \rightarrow \text{ker}(d_1)
\end{array} \]

\[ \begin{array}{c}
\text{d}_1
\end{array} \]

\[ \text{d}_2
\]

\[ \begin{array}{c}
\text{ker}(d_2)
\end{array} \]

\[ \begin{array}{c}
\text{ker}(d_1)
\end{array} \]

\[ \begin{array}{c}
\text{ker}(d_0)
\end{array} \]

\[ \text{ker}(d_0) = \mathbb{C} \]

\[ d_0(1) = 1 \]
What is \( \ker(d_k) \)? \( \ker(d_k) = 0 \)

\[
0 \leftarrow R/I \leftarrow R \leftarrow R(-3)^t \leftarrow R(-4)^3 \leftarrow 0
\]

\[
\beta_0 = 1 \quad \beta_{13} = 4 \quad \beta_{24} = 3
\]

*is a free resolution of \( R/I \)*

**Notation**

\( R(-d) \) is a rank one free \( R \)-module

It has a basis \( \{e_i\} \) where \( e_i \) has degree \( d_i \).

**Def.**

The resolution is minimal if for every differential

\( d_i : F_i \rightarrow F_{i-1} \), the image of \( d_i \) is

contained in \( M \cap F_{i-1} \), where \( M \) is the max. ideal

generated by the variables of \( R \).

**Consequence:** minimal \( \Rightarrow \) ranks are as small as possible.

**Then.**

Min. free resolutions over \( k[x_1, \ldots, x_n] \) exist and

are unique up to isomorphism.

They have length \( \leq n \). (Hilbert’s Syzygy Theorem)
Now we introduce a group action.

\[ D_5 = \langle r, s \mid r^5, s^2, sr = rs^{-1} \rangle \]

\[ \text{rotation by } 2\pi/k \]

\[ \text{reflection about a line of symmetry} \]

\[ D_5 \subseteq R = \mathbb{C}[x,y] \]

\[ \omega = e^{\frac{2\pi i}{5}} \]

5th primitive root of 1

\[ r \cdot x = \omega x, \quad r \cdot y = \omega^4 y \]

\[ s \cdot x = y, \quad s \cdot y = x \]

Elements of \( D_5 \) act on \( \langle x,y \rangle \) by matrices:

\[ r \mapsto \begin{bmatrix} \omega & 0 \\ 0 & \omega^4 \end{bmatrix}, \quad s \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

Extend the action by setting \( \forall g \in D_5 \forall p \in \mathbb{C}[x,y] \)

\[ g \cdot p(x,y) = p(g \cdot x, g \cdot y) \]

This action is \( \mathbb{C} \)-linear on \( R \) and it is compatible with multiplication (\( g \cdot (p \cdot q) = (g \cdot p)(g \cdot q) \)).
In our example, $D_5 \subseteq I \subseteq R = \langle x^3, x^2 y, x y^2, y^3 \rangle$. Hence the action of $D_5$ descends to $R/I : \bar{g} \circ \bar{p} = \bar{g} \cdot \bar{p}$. Let's extend the action of $D_5$ to the minimal free resolution from before.

\[
0 \leftarrow R/I \leftarrow R \leftarrow R(-3)^4 \leftarrow \cdots
\]

To define $\bar{r} = e_1$, look at $r = d_4(e_1)$:

\[r \cdot d_4(e_1) = r \cdot x^2 = (r \cdot x)^3 = (\omega x)^3 = \omega^2 x^3\]

So define $\bar{r} \cdot e_1 = \omega^3 e_1$. This way $r \cdot d_4(e_1) = d_4(r \cdot e_1)$.

Similarly, for other basis elements:

$r$ acts on $\langle e_2, e_3, e_4, e_5 \rangle$ by:

\[
\begin{bmatrix}
\omega^3 & 0 & 0 & 0 \\
0 & \omega & 0 & 0 \\
0 & 0 & \omega^4 & 0 \\
0 & 0 & 0 & \omega^3
\end{bmatrix},
\]

and for other group elements: $s$ acts by:

\[
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}.
\]
Now we can extend the action of $D_5$ to $E_8$ modules:

\[
\begin{bmatrix}
y & 0 & 0 \\
-x & y & 0 \\
0 & -x & y \\
0 & 0 & -x
\end{bmatrix}
\]

\[R(-3)^4 \xrightarrow{\phi_1 \phi_2 \phi_3} R(-4)^3\]

$r \cdot f_1$? Look at $r \cdot d_2(f_1)$:

\[r \cdot d_2(f_1) = r \cdot (y e_1 - x e_2) = (r \cdot y)(r \cdot e_1) - (r \cdot x)(r \cdot e_2) = \omega^4 y \omega^2 e_1 - \omega x \omega^2 e_2 = \omega^2 (y e_1 - x e_2) = \omega^2 d_2(f_1)\]

so take $r \cdot f_1 = \omega^2 f_1$. This gives $r \cdot d_2(f_1) = d_2(r \cdot f_1)$.

Kernels of action are $\langle \phi_1, \phi_2, \phi_3 \rangle e_i$:

\[
\begin{bmatrix}
\omega^2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \omega^3
\end{bmatrix},
\begin{bmatrix}
0 & 0 & -1 \\
0 & -1 & 0 \\
-1 & 0 & 0
\end{bmatrix}.
\]
Recap

\[ R = \mathbb{C}[x_1, \ldots, x_n], \ M \text{ fin. gen. graded } R\text{-module} \]

- \( C \) acts on \( R \) and \( M \).
- \( C \) is linearly
- preserving degrees
- compatible with multiplication \( [g \cdot (rm) = (g \cdot r)(g \cdot m)] \)

Under these assumptions, \( C \) acts on a minimal free resolution of \( M \). We saw an example how to compute this action.

Now we generalize.

Say \( (F_0, d_0) \) is a minimal free resolution of \( M \).
Assume we know how \( g \in G \) acts on \( F_{i-1} \).

\[ \begin{array}{ccc}
F_{i-1} & \xrightarrow{d_i} & F_i \\
\downarrow g & & \downarrow g \\
F_{i-1} & \xleftarrow{d_i} & F_i
\end{array} \]

the maps \( (g \cdot \cdot) \) are \( C \)-linear but not \( R \)-linear.

**On**

\[ 0 \leftarrow R/I \leftarrow R \]
\[ 0 \leftarrow R/I \leftarrow R \]
\[ r_0(x^2) \neq x \left[ r_0x \right] \]
\[ \omega^2 x \neq x \omega^3 \]
\[ \omega x \neq x \omega^3 \]
To “fix” this define an R-module map

\[ \phi_i^3 : F_i \longrightarrow F_i \]

that acts on a basis of \( F_i \) the same way as \( g \):

if \( e_1, \ldots, e_k \) is a \( \mathbb{R} \)-basis of \( F_i \), then define \( \phi_i^3(e_j) = g \cdot e_j \left( = \sum_{k=1}^n a_{jk} e_k \right) \), \( a_{jk} \in \mathbb{C} \).

Now I want a diagram to replace the previous one:

\[
\begin{array}{ccc}
F_{i-1} & \xleftarrow{\phi_i^3} & F_i \\
\phi_{i-1} \uparrow & & \uparrow \phi_i^3 \\
F_{i-1} & \xleftarrow{d_i} & F_i \\
\end{array}
\]

I can look at another complex:

\((F_\circ, d_\circ^3)\)

Claim \((F_\circ, d_\circ^3)\) is also a semi-free res. of \( M \),
and \( \phi_i^3 : (F_\circ, d_\circ) \rightarrow (F_\circ, d_\circ^3) \) is a map of complexes.
Look at the differential:

\[ F_{i-1} \xrightarrow{d_i} F_i \]

with respect to these bases, \( d_i \) is represented by a matrix \([a_{uv}]\) where

\[ d_i(e_v) = \sum_{u=1}^{n} a_{uv} e_u. \]

Claim: \( d_i^g \) is represented by \([g^{-1}a_{uv}]\) (same bases)

\[
d_i^g(e_v) = \phi_{x-1}^g d_i(\phi_{x-1}^g)^{-1}(e_v) = \phi_{x-1}^g d_i \phi_{x-1}^{g^{-1}}(e_v)
\]

\[
= \phi_{x-1}^g d_i (g^{-1}e_v) = \phi_{x-1}^g g^{-1} d_i(e_v) = \phi_{x-1}^g g^{-1} \left( \sum_{u=1}^{m} a_{uv} e_v \right)
\]

\[
= \phi_{x-1}^g g^{-1} \left( \sum_{u=1}^{n} g^{-1}a_{uv} e_u \right) = \phi_{x-1}^g \sum_{u=1}^{m} \left( g^{-1}a_{uv} \right)(g^{-1}e_u)
\]

\[
\sum_{u=1}^{m} (g^{-1}a_{uv}) \phi_{x-1}^g (g^{-1}e_u) = \sum_{u=1}^{m} (g^{-1}a_{uv}) e_u.
\]
Now we can get (almost) the map $\Phi_i^3$:

\[ 0 \to H \to F_0 \to F_1 \to F_2 \to \cdots \]

We can easily compute by acting with $s^{-1}$ on the matrices of $d_i$.

\[ \Phi_0^3 = \Phi_0^3 \]

\[ \Phi_1^3 \]

\[ \Phi_2^3 \]

\[ \Psi_0^3 \]

\[ \Psi_1^3 \]

\[ \Psi_2^3 \]

\[ \cdots \]

(by HZ)

Can lift $\Phi_0^3$ to a map of complexes

\[ \Psi_0^3 : (F_0, d_0) \to (F_0, d_0) \]

In principle, $\Psi_i^3 \neq \Phi_i^3$ but they induce the same map in homology.

If $F_i$ has basis $e_{i1}, \ldots, e_n$, then let

$\Psi_i^3$ and $\Phi_i^3$ act on $e_n$ by sending it to

\[ g - e_n + (\text{something in } \frac{\text{im } F_i}{\bigcup_{j<i} \text{im } F_j}) \]

\[ \langle x_{i1}, \ldots, x_n \rangle \]
A finite group $G$ acts linearly on a vector space $V$ over the field $k$.

$\chi_V : G \rightarrow k$

$g \mapsto \text{trace } (V \overset{g}{\rightarrow} V)$

$\chi_V (g \cdot g^{-1}) = \chi_V (e)$

If $\dim k = 0$ or char $k | G |$, then $V \cong W$ as $G$-rep. $\iff \chi_V = \chi_W$

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$F_*$ free resolution (graded)

Say $F_i$ has basis $\langle e_1, \ldots, e_n \rangle$

Betti numbers of $F_*$:

$\beta_{ij} := \left| \{ e_k \text{ in a basis of } F_i \mid \deg(e_k) = j \} \right|$

I call $\beta_{ij}$ Betti characters

$\beta_{ij} = \text{character of the } G \text{-rep. spanned by the } e_k \text{ of degree } j \text{ in a basis of } F_i$