Jets of graphs

(joint E. Helvik and H. Vakil)

Suppose you have a hypersurface

\[ f(x_1, \ldots, x_n) = 0 \]

\[ f \in k[x_1, \ldots, x_n] \]

Apply a substitution \( x_i \rightarrow x_i + t x_i + t^2 x_i + \cdots + t^s x_i \)

for a given \( s \in \mathbb{N} \) (\( s \) is the same for all variables).

\[ f(\ldots, - \sum_{j=0}^{s} t^j x_i, \ldots) = \sum_{j=0}^{s} f_j t^j \]

where \( f_j \in k[x_0, \ldots, x_n] \) \([\text{in } n(s+1) \text{ variables}]\).

The space of \( s \)-jets of the hypersurface is the variety defined by \( f_0 = 0, f_1 = 0, \ldots, f_s = 0 \).

Note: 0-jets is the same as the original hypersurface. 1-jets is the tangent bundle.

To construct \( s \)-jets of an affine variety \( V = \cup (f_i = 0, f_m) \)

do the same thing for each hypersurface \( f_i = 0 \) and collect all coefficients.

Given a graph (connected and simple), we can associate a squarefree monomial ideal called edge ideal.

\[ \begin{array}{c}
\text{G:} \\
\text{e:} \\
\text{c:} \\
\text{b:} \\
\end{array} \quad \begin{array}{c}
\text{a} \\
\text{b} \\
\text{c} \\
\end{array} \]

\[ I(G) = \langle ab, bc \rangle \text{ ideal in } k[a, b, c] \]
What happens when we take jets of $I(G)$?

Ex. (1-jets)

$a \mapsto a_0 + \lambda a_1$, \hspace{1em} b \mapsto b_0 + \lambda b_1, \hspace{1em} c \mapsto c_0 + \lambda c_1$

$(a_0 + \lambda a_1)(b_0 + \lambda b_1) = a_0 b_0 + \lambda (a_0 b_1 + a_1 b_0) + \lambda^2 \ldots$

$(b_0 + \lambda b_1)(c_0 + \lambda c_1) = b_0 c_0 + \lambda (b_0 c_1 + b_1 c_0) + \lambda^2 \ldots$

The ideal defining first jets of $I(G)$ is

$J_1 = \langle a_0 b_0, a_0 b_1 + a_1 b_0, b_0 c_0, b_0 c_1 + b_1 c_0 \rangle$

not a monomial ideal!

$a_0 b_1 (a_0 b_1 + a_1 b_0) \in J_1$

$a_0^2 b_1^2 + a_0 b_1 a_1 b_0 \Rightarrow a_0^2 b_1^2 \in J_1 \Rightarrow a_0 b_1 \in \sqrt{J_1}$

I-ideal

$\sqrt{I} = \{ f \mid \exists m \in \mathbb{N} : f^m \in I \}$ is an ideal called radical of $I$.

It turns out

$\sqrt{J_1} = \langle a_0 b_0, a_0 b_1, a_1 b_0, b_0 c_0, b_0 c_1, b_1 c_0 \rangle$

is a monomial ideal!

It corresponds to a graph:

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We take this to be
\[ J_1(G) \]
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let $I$ be an ideal generated by squarefree monomials in the variables $x_{i_1}, \ldots, x_n$. For every $s \in \mathbb{N}$, the radical $\sqrt{J_s(I)}$ is a squarefree monomial ideal in the variables $x_{i,j}$ for $i = 1, \ldots, n$, $j = 0, \ldots, s$. Moreover, $\sqrt{J_s(I)}$ is minimally generated by the monomials:

$$x_{i_1,j_1} \cdots x_{i_r,j_r}$$

s.t. $x_{i_1} \cdots x_{i_r}$ is a minimal generator of $I$ and

$$\sum_{k=1}^r j_k \leq s.$$

Def. Let $G$ be a graph. For $s \in \mathbb{N}$, the graph corresponding to the squarefree monomial ideal $\sqrt{J_s(I(G))}$ is called the graph of $s$-jets of $G$.

**Lemma.** Let $G$ be a graph and let $a, b$ be distinct vertices of $G$. For every $s \in \mathbb{N}$, the set $\{a, b, j\}$ is an edge in $J_s(G) \iff \{a, b, j\}$ is an edge in $G$ and $i + j \leq s$. 

**Chromatic numbers**

**Theorem.** The graph $G$ has chromatic number $c \iff \forall x \in V, T_0(G) \text{ has chromatic number } c$.

**proof.** Let $G$ has vertices $\{x_i, \ldots, x_n\}$ and $T_0(G)$ has vertices $\{x_{ij}, \ldots, x_{ij}\} | j = 0, \ldots, n$. Given a minimal coloring of $\{x_i, \ldots, x_n\}$, assign $x_{ij}$ the same color as $x_i$. This is a minimal coloring by the previous lemma and the fact that $T_0(G)$ contains $T_0(G)$ as an induced subgraph and $T_0(G) \subseteq G$. $\Box$

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**Co-chordality**

**Def.** A graph is **chordal** if it contains no induced cycle of length 4 or more.

A graph is **co-chordal** if the complement is chordal.

**Ex.**

\[
\begin{array}{c}
\text{not chordal} \\
\text{co-chordal}
\end{array}
\]

$I(C_4) = \{ab, bc, ca, ad\}$

**Theorem.** (Fröberg) $I(G)$ has a linear reduction if and only if $G$ is co-chordal.
Theorem. If a graph $G$ has diameter $\geq 3$, then
\[ \forall n \geq 1 \quad T_n(G) \text{ is not co-chordal}. \]

Diameter = max. distance between vertices.

Diameter 1 graphs are complete. We showed:

Prop. $\forall n \geq 2$ and $n > 0$, $T_n(K_n)$ is co-chordal.

I complete graph on $n$ vertices.

For diameter 2, we have evidence of different scenarios arising but we proved:

Prop. $\forall n \geq 2$ and $n > 0$, $T_n(K_{2,n})$ is co-chordal

I complete bipartite on 2 and $n$ vertices (a star).

Example: $T_5(G)$ is co-chordal (using M2 + Fröbeke)

Then $G$ is not co-chordal (by looking at parameters of edge ideal)
Def. A simplicial vertex of a graph is a vertex whose neighbors induce a complete subgraph.

Def. A simplicial order of a graph is an enumeration $x_1, \ldots, x_n$ of the vertices such that for every $i < j$, the vertex $x_i$ is simplicial for the graph induced on $x_i, \ldots, x_n$.

Thm. A graph is chordal $\iff$ it has a simplicial order.

$x_1, x_2, x_3$

$K_3$

$\mathcal{F}(K_3)$

$x_i, t < x_j, u \iff t \leq u$ or, $t = u$ and $i < j$.
**Vertex covers of jet graphs**

**Def.** A subset \( C \) of the set of vertices of a graph is called a **vertex cover** if every edge of the graph has an endpoint in \( C \). A **minimal vertex cover** is one whose proper subsets are not vertex covers.

\[ G: \quad \begin{array}{c}
\circ & \circ & \circ \\
\text{minimal:} & \{b\} & \text{one vertex covers} \\
\{a, c\} & \{a, b, c\} & \text{not minimal}
\end{array} \]

\[ I(G) = \langle ab, bc \rangle = \langle b \rangle \cap \langle a, c \rangle \]

Minimal vertex covers of \( G \) correspond to irreducible components of \( I(G) \).

\[ G: \quad \begin{array}{c}
\circ & \circ & \circ \\
\text{all the min. vertex covers:} & \{a, b, c\}, \{a, c\}, \{b, c\}
\end{array} \]

\[ J_4(K_3): \quad \begin{array}{c}
\circ & \circ & \circ \\
\{a_0, b_0, c_0\}, \{a_0, b_0, a_1, b_1, c_1\}, \{a_0, c_0, a_1, c_1\}, \{b_0, c_0, b_1, c_1\}
\end{array} \]

There are the min. vertex covers of \( J_4(K_3) \).
Prop. Let $G$ be a graph with vertices $x_1, \ldots, x_n$.

1) $\forall s \in \mathbb{N}$, the set
   \[
   \{ x_1, x_s, \ldots, x_{s+t}, x_{s+1}, \ldots, x_n, \ell \mid s, t = 0, 1, 2, \ldots \}
   \]
   is a min. vertex cover of $T_{2s+1}(G)$.

2) If $\{ x_{i_1}, \ldots, x_{i_k} \}$ is a min. vertex cover of $G$, then $\forall s \in \mathbb{N}$ the set
   \[
   \{ x_1, x_s, \ldots, x_{s+t}, x_{s+1}, \ldots, x_{i_1}, \ldots, x_{i_k}, \ell \mid s, t = 0, 1, 2, \ldots \}
   \]
   is a min. vertex cover of $T_{2s}(G)$.

Prop. Let $G$ be a graph with vertices $x_1, \ldots, x_n$.
If $\{ x_{i_1}, \ldots, x_{i_k} \}$ is a min. vertex cover of $G$, then
   \[
   \{ x_{i_1}, \ell, x_{i_2}, x_{i_3}, \ldots, x_{i_k}, x_{i_{k+1}} \mid s, t = 0, 1, 2 \}
   \]
   is a min. vertex cover of $T_{2s}(G)$. 
En. \( \{a_0, b_0, c_0, a_1, b_1, a_2, b_2\} \)

is a min. vertex cover of \( T_3(K_3) \)

Def. A graph is **well-covered** if all min. vertex covers have the same cardinality.

En. Jers of well-covered graphs are more well-covered:

\( K_3, T_3(K_3) \)

Def. A graph is **very well-covered** if it is well-covered and each min. vertex cover contains half of the vertices.

[Favaron, 1982]

En. \( K_{n,n} \) are very well-covered

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\( \begin{array}{ccc}
  x_1 & \ldots & x_3 \\
  y_1 & \ldots & y_3 \\
  x_1 & \ldots & y_1 \\
  y_1 & \ldots & x_1 \\
\end{array} \)
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\( \{x_1, x_2, x_3\} \)

\( \{y_1, y_2, y_3\} \)

min. vertex covers

Thm. \( n \geq 1, m > 0 \) \( T_3(K_{n,m}) \) is very well-covered.

Idea. Describe min. vertex covers explicitly.

Say the graph \( K_{n,m} \) has vertices \( x_{1-}, x_n \) and \( y_{1-}, y_n \).

The graph \( T_3(K_{n,m}) \) has vertices \( x_i, t \) and \( y_i, t \)

with \( i=1, \ldots, n \) and \( t=0, \ldots, m \).

For \( \rho = 0, -1, \rho+1 \)

\( C_\rho = \{x_1, t, -1, x_n, t | t=0, -1, \rho\} \)

\( U \{y_1, u, -1, y_n, u | u=0, -1, \rho-1\} \)

is a min. vertex.
Every min. vertex is \( \leq p \) for some \( p \).  

\[ \text{in } \quad \text{even cycles are not very well-covered for } n > 4 \]

\[ \text{their jets are not well-covered} \]

\[ \text{in } \quad \text{Thawron's very well-covered not Kn,n} \]

\[ \text{Conjecture. } \quad \text{jets of very well-covered graphs are very well-covered.} \]