Last time: $X$ digital image

\[ \pi_1(X; x_0) \text{ - fundamental gp. of digital } \]

\[ x : \text{Im} \to X \]

\[ \beta : \text{Im} \to X \]

\[ \partial \beta \circ \alpha \simeq \beta \circ \alpha \]

so only loops of same length can be homotopic.

Edge groups:

\[ E(K; v_0) \]

\[ K \text{ simplicial complex} \]

\[ V \text{ - vertices of } K \]

\[ E \text{ - edges of } K \]

\[ \text{(incl. repeats)} \]

Edge path/loop: \[ \{v_0, v_1, v_2, \ldots, v_n\} \text{ w/ } v_i \in V. \]

\[ s^* \{v_i, v_i\} \in E. \]

\[ (\text{or } v_i = v_0) \]

**Definition 4.1.** By an elementary edge-homotopy (relative the endpoints) we mean one of the following operations on edge paths:

(a) If \( v_i = v_{i+1}, \) for some \( i \) with \( 0 \leq i \leq n - 1, \) then replace an edge path \( \{v_0, \ldots, v_i, v_{i+1}, v_{i+2}, \ldots, v_n\} \) with \( \{v_0, \ldots, v_i, v_{i+2}, \ldots, v_n\}. \) Namely, delete a repeated vertex. Or, conversely, for any \( i \) with \( 0 \leq i \leq n, \) replace an edge path \( \{v_0, \ldots, v_i, v_{i+1}, \ldots, v_n\} \) with \( \{v_0, \ldots, v_i, v_{i+1}, \ldots, v_n\}. \) Namely, insert a repeat of a vertex.

(b) If \( \{v_{i-1}, v_i, v_{i+1}\} \) form a simplex of \( K, \) for some \( i \) with \( 1 \leq i \leq n - 1, \) replace an edge path \( \{v_0, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_n\} \) with \( \{v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\}. \) Or, conversely, for any \( i \) with \( 0 \leq i \leq n - 1, \) replace an edge path \( \{v_0, \ldots, v_i, v_{i+1}, \ldots, v_n\} \) with \( \{v_0, \ldots, v_i, v, v_{i+1}, \ldots, v_n\} \) for any \( v \in V \) for which \( \{v, v, v_{i+1}\} \) form a simplex of \( K. \)
Two edge loops are edge homotopic if and only if there is a finite sequence of (a)'s & (b)'s from $\alpha \sim \beta$.

- equivalence relation on edge loops.

- equivalence class of $\alpha$ is $[\alpha]$.

$E(k, v_0)$ - set of all such is a group.

product: $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$ if $\alpha$ then $\beta$.

inverses: $[\alpha]^{-1} = [\bar{\alpha}]$ reverse loop.

Hence, $E(k, v_0) \cong \prod_{[k, v_0]}$ (topological fundamental group).
Clique complex of a digital image (graph).

$X$ digital image: $c(x)$ - simplicial complex

Simplices $\leftrightarrow$ cliques.

E.g.

$X$: 

$cl(x)$

solid (3-simplex)

Hence

$\pi_1(X; v_0) \cong E(cl(x), v_0)$

$\cong \pi_1(|cl(x)|; v_0)$

(classical result)
Definition 5.1. Consider a set \( C = \{ x_0, x_1, \ldots, x_{n-1} \} \) of \( n \) (distinct) points in \( \mathbb{Z}^m \), with \( n \geq 4 \) and for any \( m \geq 2 \). We say that \( C \) is a circle of length \( n \) if we have adjacencies \( x_i \sim_C x_{i+1} \) for each \( 0 \leq i \leq n-2 \), and \( x_{n-1} \sim_C x_0 \), and no other adjacencies amongst the elements of \( C \).

We may parametrize a digital circle as a loop \( \alpha : I_n \to X \) (in various ways).

Theorem 5.2. \( \pi_1(C; x_0) \cong \mathbb{Z} \) for every digital circle \( C \).

Definition 5.4. Suppose \( U \) and \( V \) are digital images in some \( \mathbb{Z}^n \). Denote by \( U' = \{ v \in V \mid v \notin V \cap U \} \) the complement of \( U \) in \( U \cup V \) and by \( V' = \{ u \in U \mid u \notin U \cap V \} \) the complement of \( V \) in \( U \cup V \). We say that \( U \) and \( V \) have disconnected complements (in \( U \cup V \)) if \( U' \) and \( V' \) are disconnected from each other. That is, \( U \) and \( V \) have disconnected complements when the set of pairs \( \{ u, v \} \) with \( u \in V' \), \( v \in U' \) and \( u \sim_{U \cup V} v \) is empty.

Theorem 5.5 (Digital Seifert and Van Kampen). Let \( U \) and \( V \) be digital images in some \( \mathbb{Z}^n \) with connected intersection \( U \cap V \). Choose \( x_0 \in U \cap V \) for the basepoint of \( U \cap V, U, V, \) and \( U \cup V \). If \( U \) and \( V \) have disconnected complements, then

\[
\begin{align*}
p_1(U \cap V; x_0) & \xrightarrow{i_1} p_1(U; x_0) \\
i_2 & \downarrow \\
p_1(V; x_0) & \xrightarrow{\psi_2} p_1(U \cup V; x_0)
\end{align*}
\]

is a pushout diagram of groups and homomorphisms, with \( i_1, i_2, \psi_1 \) and \( \psi_2 \) the homomorphisms of fundamental groups induced by the inclusions \( U \cap V \to U, U \cap V \to V, U \to U \cup V \) and \( V \to U \cup V \) respectively.

\[\text{Ex. If } \pi_1(U \cup V; x_0) \cong \{ e \}, \text{ then }\]

\[
\pi_1(U \cup V; x_0) \cong \pi_1(U; x_0) \ast \pi_1(V; x_0)
\]
We have $\Pi_1(X, x_0) \cong \mathbb{Z}_2$. 

**Figure 1.** Realization of $D \vee D$ in $\mathbb{Z}^2$

**Figure 2.** Triangulation of $\mathbb{R}P^2$

Of the 13 points
- $5 = (1, 0, 1, 0, -1, -1, 0, 0)$
- $6 = (1, 1, 0, 0, 0, -1, -1, 0)$
- $7 = (0, 1, -1, -1, 0, 0, -1, 0)$
- $8 = (-1, 1, 0, -1, -1, 0, 0, 0)$
- $9 = (-1, 0, 1, 0, -1, -1, 0, 0)$
- $10 = (-1, -1, 0, 0, 0, -1, -1, 0)$
- $11 = (0, -1, -1, -1, 0, 0, 0, 0)$
- $12 = (1, -1, 0, -1, -1, 0, 0, 0)$
- $13 = (0, 0, 0, 1, 0, -1, -1, 0)$
- $3 = (0, 0, 0, 0, 0, 1, 0, -1)$
- $4 = (0, 0, 0, 0, 0, 0, 1, -1)$